Math 116b - Homework 8

`Instructor:` Andrés Eduardo Caicedo

`Due:` March 11, 2008 at 1:00 pm.

This Homework is due **at the beginning of lecture** by Tuesday March 11 at 1:00 pm. Refer to the grading policy for additional requirements.

1. Argue that if $A, B \subseteq \mathbb{N}$, then $A \leq_T B$ iff A is Δ_1 in B. Use this to show:
   (a) Let $S \subseteq \mathcal{P}(\mathbb{N})$. Then $(\mathbb{N}, S, +, \times, 0, 1, <) \models \mathsf{RCA}_0$ iff $S \neq \emptyset$, $A, B \in S$ implies $A \oplus B \in S$, and $A \in S$ and $B \leq_T A$ imply $B \in S$.
   (b) Let $S \subseteq \mathcal{P}(\mathbb{N})$. Then $(\mathbb{N}, S, +, \times, 0, 1, <) \models \mathsf{ACA}_0$ iff in addition to the above, $A \in S$ implies $K_A \in S$.

2. Prove the uniform recursion theorem: If $f(x, y)$ is total recursive, then there is a total recursive unary g such that for all x,

   $$\varphi_{f(x,g(x))} = \varphi_{g(x)}.$$

   Also, prove the double recursion theorem: If $f(x, y)$ is total recursive, then there is a pair (a, b) such that $\varphi_a = \varphi_{\beta_2^1(f(a,b))}$ and $\varphi_b = \varphi_{\beta_2^2(f(a,b))}$.

3. Say that $A \subseteq \mathbb{N}$ is Π_2 complete iff A is Π_2 (i.e., definable by a formula of the form $\forall \vec{x} \theta(\vec{x}, y)$, where θ is Σ_1) and for any Π_2 set B, $B \leq_m A$. Show that the following are Π_2 complete:
   (a) $\{x : \varphi_x \text{ is total}\}$.
   (b) $\{x : \operatorname{dom} \varphi_x \text{ is infinite}\}$.
   (c) $\{\alpha_2(x, y) : \varphi_x = \varphi_y\}$.

4. Say that A is Σ_3 iff A is definable by a formula of the form $\exists \vec{y} \psi(x, \vec{y})$ where ψ is Π_2. Say that A is Σ_3 complete iff A is Σ_3 and $B \leq_m A$ for any Σ_3 set B. Show that the following are Σ_3 complete:
   (a) $\{x : U_x^1 \text{ is finite}\}$.
   (b) $\{x : \operatorname{dom} \varphi_x \text{ is recursive}\}$.