Math 117b - Homework 2

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Due: January 18, 2007 at 1:00 pm

This Homework is due either during lecture or in the course box outside 253 Sloan by Thursday January 18 at 1:00 pm. Refer to the grading policy for additional requirements.

1. (**Kleene-Post**)

   (a) Show that there are incomparable degrees $\mathbf{a}, \mathbf{b} \leq_T \mathbf{0}'$.
   [Hint: Check that the construction given in class can be organized recursively in $\mathbf{0}'$. Explain how this implies the result.]

   (b) Show that for any degree $\mathbf{d}$ there are incomparable degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{d} \leq \mathbf{a}, \mathbf{b}$ and $\mathbf{a}, \mathbf{b} \leq \mathbf{d}'$.

2. Show the **Friedberg Jump inversion theorem**: Given $\mathbf{x} \geq_T \mathbf{0}'$ there is $\mathbf{y}$ such that $\mathbf{x} = \mathbf{y}'$.

   [Hint: Fix $X : \mathbb{N} \to 2$, $X \in \mathbf{x}$ (show that there is such an X). We will build $Y : \mathbb{N} \to 2$ by finite initial segments as $Y = \bigcup_n \sigma_n$ with $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \dots$ and each σ_i defined on a finite initial segment of $\mathbb{N}$, such that $Y' \equiv_T X$. At step 0 let $\mathrm{dom}\,(\sigma_0) = \{0\}$ and $\sigma_0(0) = 0$. At stage n, given σ_n, let l be the largest element of its domain. Define

   $$\tau = \sigma_n \cup \{(l+1, X(n+1))\}.$$

   Then proceed to "decide" the n^{th} element of Y': If there is $\sigma \supseteq \tau$ such that $\varphi_n^{\sigma}(n) \downarrow$, let σ_{n+1} be such σ. Otherwise let $\sigma_{n+1} = \tau$. This completes stage n.

   Show that this Y works, i.e., $Y' \equiv_T Y \oplus 0' \equiv_T X$.]

3. (**Sacks**)

   (a) Show that there is a set S of independent degrees with $|S| = \mathfrak{c}$, as follows: Recall that $\mathfrak{c} = 2^{\aleph_0}$, i.e., if C is the set of all (total) functions $f : \mathbb{N} \to \{0,1\}$, then $|C| = \mathfrak{c}$ (you can assume this fact).
   Consider the notion of forcing $\mathbb{P}$ where conditions are partial functions $T \vdots \{0,1\}^{<\infty} \to \{0,1\}^{<\infty}$ whose domain is

   $$\{\sigma \in \{0,1\}^{<\infty} : \mathrm{lh}\,(\sigma) \leq i\}$$

   for some $i \in \mathbb{N}$ and such that

- $\forall \sigma, \tau \in \mathrm{dom}\,(T)\,(\sigma \subseteq \tau \to T(\sigma) \subseteq T(\tau))$.
- $\forall \sigma, \tau \in \mathrm{dom}\,(T)\,(\sigma \perp \tau \to T(\sigma) \perp T(\tau))$.

The order is reverse inclusion, i.e., T extends S iff $T \supseteq S$.

Show that there is a countable set $\mathcal{C}_1$ of dense sets such that if G is $\mathcal{C}_1$-generic and $A_G = \bigcup\{T : T \in G\}$ then $A_G : \{0,1\}^{<\infty} \to \{0,1\}^{<\infty}$ (i.e., $\mathrm{dom}\,(A_G) = \{0,1\}^{<\infty}$).

Let

$$\mathcal{F}_G = \{A_G(x) : x \in \{0,1\}^{\mathbb{N}}\}$$

where, for each $x : \mathbb{N} \to \{0,1\}$,

$$A_G(x) = \bigcup\{A_G(\sigma) : \sigma \subset x\}.$$

Show that $|\mathcal{F}| = \mathfrak{c}$.

We want to add dense sets to $\mathcal{C}_1$ to form a larger (but still countable) set $\mathcal{C}$ such that if G is $\mathcal{C}$-generic and $\mathcal{F}_G$ is as above, then $\mathcal{F}_G$ is an independent family.

For each $e \in \mathbb{N}$, $\sigma \in \{0,1\}^{<\infty}$ and $F = \{\tau_1, \ldots, \tau_n\}$ such that

$$F \subseteq \{\tau \in \{0,1\}^{<\infty} : \mathrm{lh}\,(\sigma) = \mathrm{lh}\,(\tau) \text{ and } \tau \neq \sigma\},$$

let $R_{e,\sigma,F}$ be the following requirement:

$$\forall A_1, \ldots, A_n, B, C\,\Big(\begin{array}{l} T(\sigma) \subseteq B, T(\tau_1) \subseteq A_1, \ldots, T(\tau_n) \subseteq A_n \text{ and} \\ C = \bigoplus_{i=1}^{n} A_i \longrightarrow \varphi_e^C \neq B. \end{array}\Big)$$

Show that it suffices to satisfy all such requirements. Define forcing for these requirements and prove the density and satisfaction lemmas. Complete the proof.

(b) Let $\mathcal{U} = (U, \leq)$ be a partial order such that $|U| \leq \mathfrak{c}$ and each member of U has only finitely many predecessors. Show that $\mathcal{U}$ embeds into $\mathcal{D}$.