Math 116b - Homework 6

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`Due:` February 26, 2008 at 1:00 pm.

This Homework is due during lecture by Tuesday February 26 at 1:00 pm. Refer to the grading policy for additional requirements.

Using the description given in lecture of a (provable in PA) truth predicate for Σ_1 statements, one can show the following result:

Theorem 1. *For all countable non-standard models $M \models \mathsf{PA}$ and for all $a \in M$ there is a proper initial segment I of M such that $a \in I$ and I is isomorphic to M; moreover, one can choose the isomorphism $f : I \to M$ such that $f \restriction [0, a]$ is the identity.*

You can assume this theorem in what follows. For a (significant) bonus problem, feel free to provide a proof.

The goal of this homework set is to present some features of the proof given in lecture of the Kanamori-McAloon independence result in a more general context. Begin by showing the following (trivial) observation: If all countable non-standard models of PA are models of ϕ, then $\mathsf{PA} \vdash \phi$.

Definition 2. Let $M \models \mathsf{PA}$ be countable and non-standard. Let $\chi \subseteq \mathcal{P}(M)$ be a collection of initial segments of M. (Neither χ nor its elements need be definable in M.)

A Σ_1 formula $\varphi(x, y, z)$ is an *indicator* for χ in M iff

1. φ defines the graph of a (total) function $Y : M^2 \to M$. In detail, this means that for some such Y, if
$$\Gamma_Y = \{(a, b, c) : a, b \in M \text{ and } c = Y(a, b)\},$$
then
$$\{(a, b, c) : M \models \varphi(a, b, c)\} = \Gamma_Y.$$
2. For each $a, b \in M$ with $a < b$, $Y(a, b)$ is non-standard iff there is $I \in \chi$ such that $a \in I$ and $b \notin I$. In the terminology introduced in lecture, $a \leq I < b$.
3. For $a, b, c, d \in M$ such that $c \leq a \leq b \leq d$, $Y(a, b) \leq Y(c, d)$.

We say that φ is an indicator for the theory T in M iff φ is an indicator for $\{I : I \text{ is an initial segment of } M \text{ and } I \models T\}$ in M. Finally, we say that φ is an indicator for the theory T in the theory $T' \supseteq \mathsf{PA}$ iff for all countable nonstandard $M \models T'$, φ is an indicator for T in M.

Let $T \supseteq \mathsf{PA}$. Let φ be an indicator for T in T, and write $Y(x, y) = z$ for $\varphi(x, y, z)$.

1. Show that $T \nvdash \forall x, z \exists y > x\, (Y(x, y) \geq z)$.
2. Show that for all $n \in \mathbb{N}$, $T \vdash \forall x \exists y > x\, (Y(x, y) \geq \underline{n})$. Conclude that if T is true (i.e., if $\mathbb{N} \models T$), then the Π_2 sentence $\forall x, z \exists y > x\, (Y(x, y) \geq z)$ is neither provable nor refutable in T.
3. For $n \in \mathbb{N}$ let $g_n(x) = \mu y\, (Y(x, y) \geq n)$. Let f be any (total) recursive function that T proves is total. Show that there is an n such that $T \vdash \forall x\, (f(x) < g_n(x))$.
4. Show that the Kanamori-McAloon result that the version of Ramsey's theorem for regressive functions is not provable in PA can be stated as a particular case of this setting by describing the indicator for PA in PA used in the proof given in lecture.