Math 116b - Homework 5

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`Due:` February 19, 2008 at 1:00 pm.

This Homework is due during lecture by Tuesday February 19 at 1:00 pm. Refer to the grading policy for additional requirements.

In second order arithmetic, a Σ_1 formula is one of the form $\exists n_1 \dots \exists n_k \, \psi$ where all quantifiers in ψ are bounded, i.e., of the form $\forall m < \tau$ or $\exists m < \tau$ where τ is some arithmetical term. (This expands the concept from first order arithmetic in that set variables are allowed in ψ, either as free variables, or as part of some atomic sub-formula $\tau \in X$.) Similarly, one can expand the notions of Π_1 and Δ_1.

Let RCA_0 be the fragment of second order arithmetic whose axioms are the basic axioms, induction for Σ_1 formulas, and the scheme of Δ_1 comprehension, which consists of all axioms of the form (the universal closure of)

$$\forall n \, (\phi(n) \leftrightarrow \psi(n)) \to \exists X \, \forall n \, (n \in X \leftrightarrow \phi(n))$$

where ϕ is Σ_1 (possibly with set and number variables), ψ is Π_1, and X is not free in ϕ.

1. Define (in RCA_0) the cross product $X \times Y$ as the set of all k such that there are $i \in X$ and $j \in Y$ such that $k = \alpha_2(i, j)$. Check that RCA_0 indeed proves that this is a set (if X and Y are), and that one can define functions in RCA_0. Check that RCA_0 proves the existence of all primitive recursive functions.

2. Show that $(\mathbb{N}, S, +, \times, 0, 1, <)$ is a model of RCA_0, where $S = \{A \subseteq \mathbb{N} : A$ is recursive$\}$.

3. Show that RCA_0 proves the *strong* Σ_1 *bounding scheme*, consisting of all axioms of the form (the universal closure of)

$$\forall m \, \exists n \, \forall i < m \, ((\exists j \, \varphi(i, j)) \to (\exists j < n) \varphi(i, j))$$

   where φ is Σ_1 and n is not free in φ.

4. Show in RCA_0 that the following are equivalent:
   (a) ACA_0.
   (b) Σ_1 comprehension, i.e., all formulas of the form (the universal closure of) $\exists X \, \forall n \, (n \in X \leftrightarrow \varphi(n))$, where φ is Σ_1 and X is not free in φ.
   (c) For all injective $f : \mathbb{N} \to \mathbb{N}$, $\mathrm{ran}(f) = \{n : \exists m \, (f(m) = n)\}$ exists.