Math 116b - Homework 3

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Due: February 5, 2007 at 1:00 pm.

This Homework is due during lecture by Tuesday February 5 at 1:00 pm. Refer to the grading policy for additional requirements.

1. The goal of this exercise is to show that Ackermann's function is recursive by showing it has a Σ_1-graph.

   The idea is that to evaluate $A(n, m)$ we only need to know a finite (but probably very long) list of values $A(l, s)$ with either $l < n$ or else $l = n$ and $s < m$.

   Accordingly, the formula we want will say that there is a number u coding such a sequence of values. Of course, we need to keep track not only of the values $A(l, s)$ but also of the numbers l and s, so we will let each $(u)_i$ code a *triple*, with the intended meaning that $(u)_i = \alpha_3(a, b, c)$ iff $A(a, b) = c$. (Recall that α_3 is a primitive recursive coding of triples by numbers.)

   Hence, our formula $\theta(x, y, z)$ could say something like "There is l" (intended to represent the length of the sequence) "such that there is u such that there is $i < l$ such that $(u)_i = \alpha_3(x, y, z)$ and, for all $i < l$, if for some $v, w \leq u$ we have $(u)_i = \alpha_3(0, v, w)$ then $w = v+1$, and if for some $v, w \leq u$ we have $(u)_i = \alpha_3(v + 1, 0, w)$, then for some $j < i$, $(u)_j = \alpha_3(v, 1, w)$" etc.

   Complete this sketch.

2. Show that a set $A \subseteq \mathbb{N}$ is r.e. iff it is empty or the range of a recursive function. Here is a possible way of doing this, but feel free to try a different approach:

   Say $A \neq \emptyset$ is r.e., so it is the domain of a partial recursive function f. Since $f \in \mathsf{R}$, it has a Σ_1 graph, say

   $$f(n) \downarrow= m \text{ iff } \mathbb{N} \models \exists z\, \theta(n, m, z)$$

   where θ is Δ_0.

   Let a_0 be a fixed element of A. Define h by

   $$h(n) = \begin{cases} \beta_2^1(n) & \text{if } \exists z, y < \beta_2^2(n)\, \theta(\beta_2^1(n), y, z) \\ a_0 & \text{otherwise.} \end{cases}$$

   Show that this works and that, in fact, h so defined is primitive recursive.

3. (Craig's trick) Suppose that T is an r.e. set of sentences. We want to show that there is a *primitive recursive* set T_0 (i.e., $\chi_{T_0} \in \mathsf{PR}$) such that $\mathrm{Th}(T) = \mathrm{Th}(T_0)$. For this, let f be a primitive recursive function such that $T = \mathrm{ran}(f)$. Say $f(n) = \ulcorner \varphi_n \urcorner$. Let

$$T_0 = \{\ulcorner \varphi_n \wedge \cdots \wedge \varphi_n \urcorner : n \in \mathbb{N} \text{ and } \varphi_n \text{ appears } n \text{ times}\}.$$

Show that T_0 works.

4. In reasonable amount of detail explain how to define a Gödel numbering. You do not need to use the coding suggested in class, and you do not need to follow the approach in class if a different one works better, but you do need to be more careful than the brief sketch shown in lecture.