Math 116b - Homework 1

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`Due:` January 22, 2007 at 1:00 pm.

This Homework is due during lecture by Tuesday January 22 at 1:00 pm. I believe problem 4 may be slightly harder than the others, but I will be happy to be proven wrong. Refer to the grading policy for additional requirements.

1. Show that there are $2^{\aleph_0} = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ non-isomorphic countable models of $Th(\mathbb{N})$. Recall the language here is $\mathcal{L}_1 = \{+, \times, 0, 1, <\}$.

2. Let $M = (|M|, +_M, \times_M, 0_M, 1_M, <_M)$ be a model of PA. Let S_M be the collection of subsets of $|M|$ that are arithmetically definable over M. Show that $M' = (|M|, S_M, +_M, \times_M, 0_M, 1_M, <_M)$ is a model of ACA_0. Conclude that if ϕ is a first-order sentence in the language of arithmetic and $\mathsf{ACA}_0 \vdash \phi$ then $\mathsf{PA} \vdash \phi$, i.e., ACA_0 *is conservative over* PA.

3. Let $\mathbb{Z}[X]^+$ be the $\mathcal{L}_1$-structure of all non-negative polynomials in the ring $\mathbb{Z}[X]$. Here, given $a_0, \dots, a_n \in \mathbb{Z}$ with $a_n \neq 0$, we say that $a_0 + a_1 X + \cdots + a_n X^n > 0$ iff $a_n > 0$ and in general given $p, q \in \mathbb{Z}[X]$, we say that $p > q$ iff $p - q > 0$.

   Show that $\mathbb{Z}[X]^+ \models Q$, Robinson's arithmetic, but that $\mathbb{Z}[X]^+$ is not a model of $Th(\mathbb{N})$ (it is not even a model of PA).

4. $(*)$ Let $M \models \mathsf{PA}$. A number $n \in M$ is *non-standard* iff for all $m \in \mathbb{N}$, $M \models m < n$. Show that if $M \not\cong \mathbb{N}$ then M contains non-standard numbers. Let a be one of them. Show that the map

   $$f : \mathbb{Z}[X]^+ \to M$$

   given by $f(p) = p(a)$ is a well-defined embedding (i.e., injective homomorphism) of $\mathcal{L}_1$-structures.