

# Math 117b - Homework 3

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Due: January 25, 2007 at 1:00 pm.

This Homework is due either during lecture or in the course box outside 253 Sloan by Thursday January 25 at 1:00 pm. Refer to the grading policy for additional requirements.

**(Sacks)** The goal of this exercise is to show that above any degree  $\mathbf{a}$  there is a *minimal cover*  $\mathbf{b}$ , i.e.,

$$\mathbf{a} <_T \mathbf{b}$$

and

there is no  $\mathbf{c}$  such that  $\mathbf{a} <_T \mathbf{c} <_T \mathbf{b}$ .

This will be accomplished by forcing. To avoid unnecessary complications, I will talk about dense sets directly, avoiding any mention of  $\Vdash$ . We will need some definitions.

Let  $2^{<\mathbb{N}}$  denote the collection of all finite strings of 0's and 1's. Any such a string is a function whose domain is a finite initial segment of  $\mathbb{N}$ . Let  $\sigma, \tau$  denote elements of  $2^{<\mathbb{N}}$ . By  $\sigma \frown \tau$  we mean the *concatenation* of  $\sigma$  and  $\tau$ . For example, if  $\sigma = (0, 1, 1, 0)$ —i.e.,  $\sigma$  is the function with domain  $\{0, 1, 2, 3\}$  such that  $\sigma(0) = 0$ ,  $\sigma(1) = 1$ ,  $\sigma(2) = 1$  and  $\sigma(3) = 0$ —and  $\tau = (0, 1)$ , then  $\sigma \frown \tau = (0, 1, 1, 0, 0, 1)$ .

Let  $2^{\mathbb{N}}$  denote all infinite sequences of 0's and 1's (all *characteristic functions* of subsets of  $\mathbb{N}$ ).

$\sigma$  and  $\tau$  are *compatible* iff one is an initial segment of the other; otherwise they are *incompatible*. Write  $\sigma \perp \tau$  to indicate that  $\sigma$  and  $\tau$  are incompatible.

For the purposes of this exercise, a *tree* is a subset  $T$  of  $2^{<\mathbb{N}}$  such that

- For any  $\sigma \in T$ , if  $\tau \subseteq \sigma$  then  $\tau \in T$ , and
- For any  $\sigma \in T$  there are  $\tau \perp \rho$  in  $T$  such that  $\sigma \subseteq \tau$  and  $\sigma \subseteq \rho$ .

If  $T, S$  are trees we say that  $S \leq T$  iff  $S$  is a subtree of  $T$ , i.e., iff  $S \subseteq T$ . The *full binary tree* is  $2^{<\mathbb{N}}$ . Any tree is a subtree of the full binary tree.

Since there are recursive bijections between  $\mathbb{N}$  and  $2^{<\mathbb{N}}$ , we can talk about a tree being recursive, or having degree  $\mathbf{a}$ , etc.

If  $T$  is a tree,  $[T]$  denotes the set of its *branches*, i.e., the set of  $x = (x(0), x(1), \dots) \in 2^{\mathbb{N}}$  such that for all  $n$ ,  $x \upharpoonright n = (x(m) : m < n) \in T$ . Obviously, if  $S \leq T$  then  $[S] \subseteq [T]$ . Do *not* confuse the notation  $[A]$  with  $[A]_T$ , the Turing degree of  $A$ .

Let  $T$  be a tree. The longest string  $\sigma \in T$  such that *any* string  $\tau$  in  $T$  is compatible with  $\sigma$  is called the *stem* of  $T$ . For example, the stem of the full binary tree is the empty string  $\emptyset$ . Let  $T$  be a tree and  $\tau \in T$ .  $T_\tau$ , the *full subtree of  $T$  above  $\tau$* , is  $\{\rho \in T : \tau \subseteq \rho \text{ or } \rho \subseteq \tau\}$ . Notice that  $\tau$  is an initial segment of the stem of  $T_\tau$ .

A tree  $T$  is *recursively pointed* iff  $T \leq_T A$  for any  $A \in [T]$ , i.e.,  $T$  is recursive in any of its branches. For example, any recursive tree (like the full binary tree) is recursively pointed.

1. Let  $T$  be recursively pointed. Show that  $T$  has branches of any degree above the degree of  $T$ .

[**Hint:** Let  $B \in 2^{\mathbb{N}}$ ,  $T \leq_T B$ . Define a branch  $A$  of  $T$  as follows: Let  $\sigma_0 = \emptyset$ . At stage  $n$ , some initial segment  $\sigma_n \in T$  of  $A$  has been defined. Let  $\tau$  be the stem of  $T_{\sigma_n}$ , so both  $\tau \frown (0)$  and  $\tau \frown (1)$  are in  $T$ . Let  $\sigma_{n+1} = \tau \frown (B(n))$ . Show that  $A \equiv_T B$ .]

A tree  $T$  is *over  $A$*  iff  $A \leq_T T$ .

The *left-most branch* of  $T$  is the unique  $A \in [T]$  (so  $A : \mathbb{N} \rightarrow 2$ ) such that for any  $n$  if  $(A \upharpoonright n) \frown (0)$  and  $(A \upharpoonright n) \frown (1)$  are both in  $T$ , then  $A \upharpoonright (n+1) = (A \upharpoonright n) \frown (0)$ . Notice that if  $A$  is the left-most branch of  $T$ , then  $A \leq_T T$ .

Let  $\mathbf{a}$  be a given Turing degree. Suppose that there is a recursively pointed tree  $T_1$  of degree  $\mathbf{a}$ . Our forcing notion  $\mathbb{P}$  is the collection of all recursively pointed trees  $T$  of degree  $\mathbf{a}$  that are subtrees of  $T_1$ , ordered by being a subtree:  $T \leq S$  iff  $T$  is a subset of  $S$ . (So  $T_1$  is the maximum element of  $\mathbb{P}$ .)

If  $G$  is a filter, then any two trees  $T, S \in G$  are *compatible*, which means that  $T \cap S$  is a tree (and is in  $G$ ). Notice that this is a different notion from the notion of compatibility of strings.

Let  $T_G = \bigcap \{T : T \in G\}$  and  $A_G = \bigcup \{\sigma : \sigma \in T_G\}$ . Notice that  $T_G$  is *not* necessarily a tree. We will find a family of dense sets  $\mathcal{C}$  such that if  $G$  is  $\mathcal{C}$ -generic, then  $A_G \in 2^{\mathbb{N}}$ —so  $A_G$  is a branch through every tree in  $G$ , and it is the unique such branch.

2. Let  $T$  be recursively pointed. Let  $T \leq_T A$ . Show that there is  $S \leq T$  such that  $S$  is recursively pointed and  $S \equiv_T A$ . Conclude that there is a tree  $T_1$  as above and therefore  $\mathbb{P}$  is nonempty, no matter what  $\mathbf{a}$  is.

[**Hint:** We have  $A \in 2^{\mathbb{N}}$ . Define  $S$  by induction: Given  $\rho \in 2^{<\mathbb{N}}$  we will identify some nodes  $p_\rho \in T$  and let  $S$  be the subtree of  $T$  they generate, i.e., the set of all initial segments of these nodes. This identification is done by induction on the length of  $\rho$ .

For  $\rho = \emptyset$ , let  $\tau$  be the stem of  $T$ . Set  $p_\emptyset = \tau \frown (0)$  if  $A(0) = 0$ . Otherwise, set  $p_\emptyset = \tau \frown (1)$ .

Having defined  $p_\rho$  for all  $\rho$  of length  $n$ , we want to define  $p_{\rho \frown (0)}$  and  $p_{\rho \frown (1)}$ . Let  $\sigma$  be the stem of  $T_{p_\rho}$ . Let  $\sigma^0$  be the stem of  $T_{\sigma \frown (0)}$  and  $\sigma^1$  be the stem of  $T_{\sigma \frown (1)}$ . Let  $i = A(n)$ , so  $i \in \{0, 1\}$ . Set  $p_{\rho \frown (0)} = \sigma^{i \frown (0)}$  and  $p_{\rho \frown (1)} = \sigma^{i \frown (1)}$ .

Verify that  $S \leq_T T \oplus A \equiv_T A$ . Let  $B \in [S]$ . Show that  $A \leq_T T \oplus B$  and therefore  $A \leq_T B$ . Consider the left-most branch of  $S$  to conclude that  $A \leq_T S$ . Show that  $S \leq_T T \oplus B$  and conclude that  $S$  is recursively pointed.]

3. Let  $T$  be recursively pointed. Let  $S \leq T$  be such that  $S \leq_T T$ . Then  $S$  is recursively pointed and  $S \equiv_T T$ .
4. Let  $C_n = \{T \in \mathbb{P} : \text{the stem of } T \text{ has length at least } n\}$ . Show that each  $C_n$  is dense and therefore if  $G$  is  $\mathcal{C}$ -generic, where  $C_n \in \mathcal{C}$  for all  $n$ , then  $A_G$  is the unique branch through all the trees in  $G$ . Conclude that  $\mathbf{a} \leq_T A_G$ .

By 4., to ensure that  $\mathbf{a} <_T A_G$ , it suffices to ensure that  $A_G \not\leq_T \mathbf{a}$ . Fix  $B \in \mathbf{a}$ .

5. For  $e \in \mathbb{N}$  let  $D_e = \{T \in \mathbb{P} : \text{for all } A \in [T], A \neq \varphi_e^B\}$ . Show that each  $D_e$  is dense. Conclude that if, in addition to the sets in 4., each  $D_e \in \mathcal{C}$  and  $G$  is  $\mathcal{C}$ -generic, then  $\mathbf{a} <_T A_G$ .

[Hint: Given a tree  $T \in \mathbb{P}$ , let  $\tau$  be its stem and consider  $\tau \frown (0)$  and  $\tau \frown (1)$ . Since they are incompatible, at least one must disagree with  $\varphi_e^B$ . Let  $S$  be the full subtree of  $T$  above it. Check that  $S \in D_e$ .]

Now we need to ensure that there is no  $B$  such that  $\mathbf{a} <_T B <_T A_G$ . Any  $B \leq_T A_G$  has (by definition) the form  $\varphi_e^{A_G}$  for some  $e$ . Given  $e \in \mathbb{N}$ , fix  $E \in \mathbf{a}$  and suppose we manage to ensure that one of the following cases holds:

- $\varphi_e^{A_G}$  is not total.
  - $\varphi_e^{A_G} \leq_T E$ .
  - $A_G \leq_T \varphi_e^{A_G} \oplus E$ .
6. If the situation above happens, conclude that  $\varphi_e^{A_G}$  is not a characteristic function of a set of degree intermediate between  $E$  and  $A_G$ .

We now proceed to verify that we can indeed fall into one of these three situations.

7. Given  $e \in \mathbb{N}$  let  $H_e = \{T \in \mathbb{P} : \text{one of the following holds:}$

- (a) For all  $A \in [T]$ ,  $\varphi_e^A$  is not total.
- (b) For all  $A \in [T]$ ,  $\varphi_e^A \leq_T T$ .
- (c) For all  $A \in [T]$ ,  $A \leq_T \varphi_e^A \oplus T$ .

Assume each  $H_e$  is dense. Explain how to conclude the desired result.

8. Show that each  $H_e$  is dense. For this, fix  $e$  and  $T$ . Assume (\*): There is  $\sigma \in T$  and some  $x \in \mathbb{N}$  such that for any  $\tau \supseteq \sigma$ , if  $\tau \in T$  then  $\varphi_e^\tau(x) \uparrow$ . Show that in this case there is an extension of  $T$  in  $H_e$  for which clause (a) holds.

From now on assume (\*) fails. We define  $S \leq T$ ,  $S \in \mathbb{P}$  by defining nodes  $p_\sigma$  for each  $\sigma \in 2^{<\mathbb{N}}$  and then setting  $S$  to be the set of all initial segments of these nodes. Let  $p_\emptyset$  be the least  $\tau \in T$  such that  $\varphi_e^\tau(0) \downarrow$  and this computation takes at most as many steps as the length of  $\tau$  (check that there is such  $\tau$ ). Given  $p_\sigma$ , let  $n$  be the length of  $\sigma$  and let  $\tau \supseteq p_\sigma$  be least such that  $\tau \in T$ ,  $\tau \hat{\ } (0)$  and  $\tau \hat{\ } (1) \in T$ ,  $\varphi_e^\tau(n+1) \downarrow$  and this computation takes at most as many steps as the length of  $\tau$  (check that there is such  $\tau$ ). Set  $p_{\sigma \hat{\ } (0)} = \tau \hat{\ } (0)$  and  $p_{\sigma \hat{\ } (1)} = \tau \hat{\ } (1)$ .

Show that if  $S$  is so defined,  $S \leq_T T$  and therefore  $S \in \mathbb{P}$ . Notice that for any  $A \in [S]$ ,  $\varphi_e^A$  is total.

Let (\*\*) be the following statement: For every  $\sigma \in S$  there is  $x \in \mathbb{N}$  and there are  $\tau_1, \tau_2 \in S$  extending  $\sigma$  such that  $\varphi_e^{\tau_1}(x) \downarrow \neq \varphi_e^{\tau_2}(x) \downarrow$ .

Assume (\*\*) fails and show that there is an extension of  $S$  in  $H_e$  for which clause (b) holds.

Finally, assume (\*\*) holds. We define a subtree  $Q$  of  $S$ ,  $Q \in \mathbb{P}$ , by defining nodes  $q_\sigma \in S$  for each  $\sigma \in 2^{<\mathbb{N}}$  and then setting  $Q$  to be the set of all initial segments of these nodes.  $q_\emptyset = \emptyset$ . Given  $q_\sigma$ , look for  $x \in \mathbb{N}$  and  $\tau_1, \tau_2 \supseteq q_\sigma$  in  $S$  such that  $\varphi_e^{\tau_1}(x) \downarrow \neq \varphi_e^{\tau_2}(x) \downarrow$ , and set  $q_{\sigma \hat{\ } (i)} = \tau_i$  for  $i = 0, 1$ . Verify that we can do this recursively in  $S$  and therefore  $Q \in \mathbb{P}$ . Verify that  $Q$  is in  $H_e$  as witnessed by clause (c). This completes the proof.

Using essentially the same argument one can show the following result, also due to Sacks: Every countable set of degrees has a minimal upper bound. (Warning: This is not the same as claiming that every countable set of degrees admits a *least* upper bound.) If you want, as an extra credit problem, explain how to modify the argument above to show this result.